

Characteristic subspaces and hyperinvariant frames

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Abstract

Let f be an endomorphism of a finite dimensional vector space V over a field K . An f -invariant subspace is called hyperinvariant (respectively characteristic) if it is invariant under all endomorphisms (respectively automorphisms) that commute with f . We assume $|K| = 2$, since all characteristic subspaces are hyperinvariant if $|K| > 2$. The hyperinvariant hull W^h of a subspace W of V is defined to be the smallest hyperinvariant subspace of V that contains W , the hyperinvariant kernel W_H of W is the largest hyperinvariant subspace of V that is contained in W , and the pair (W_H, W^h) is the hyperinvariant frame of W . In this paper we study hyperinvariant frames of characteristic non-hyperinvariant subspaces W . We show that all invariant subspaces in the interval $[W_H, W^h]$ are characteristic. We use this result for the construction of characteristic non-hyperinvariant subspaces.

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1 Introduction

Let V be an n -dimensional vector space over a field K and let $f : V \rightarrow V$ be K -linear. A subspace $X \subseteq V$ is said to be *hyperinvariant* under f (see e.g. [12, p. 305]) if it remains invariant under all endomorphisms of V that commute with f . If X is an f -invariant subspace of V and if X is invariant under all automorphisms of V that commute with f , then [1] we say that X is *characteristic* (with respect to f). Let $\text{Inv}(V, f)$, $\text{Hinv}(V, f)$, and $\text{Chinv}(V, f)$ be sets of invariant, hyperinvariant and characteristic subspaces of V , respectively. These sets are lattices (with respect to set inclusion), and

$$\text{Hinv}(V, f) \subseteq \text{Chinv}(V, f) \subseteq \text{Inv}(V, f).$$

The structure of the lattice $\text{Hinv}(V, f)$ is well understood ([16], [9], [17], [12, p.306]). If f is nilpotent then $\text{Hinv}(V, f)$ is the sublattice of $\text{Inv}(V, f)$ generated by

$$\text{Ker } f^k, \text{ Im } f^k, k = 0, 1, \dots, n.$$

It is known ([20], [13, p.63/64], [1]) that each characteristic subspace is hyperinvariant if $|K| > 2$. Hence, only if V is a vector space over the field $K = GF(2)$ there may exist K -endomorphisms f of V with characteristic subspaces that are not hyperinvariant.

If the characteristic polynomial of f splits over K (such that all eigenvalues of f are in K) then one can restrict the study of hyperinvariant and of characteristic subspaces to the case where f has only one eigenvalue, and therefore to the case where f is nilpotent. Thus, throughout this paper we shall assume $f^n = 0$. Let $\Sigma(\lambda) = \text{diag}(1, \dots, 1, \lambda^{t_1}, \dots, \lambda^{t_m}) \in K^{n \times n}[\lambda]$ be the Smith normal form of f such that $t_1 + \dots + t_m = n$. We say that λ^{t_j} is an *unrepeated* elementary divisor of f if it appears exactly once in $\Sigma(\lambda)$. We note the following result which is due to Shoda (see also [5, Theorem 9, p. 510] and [13, p. 63/64]).

Theorem 1.1. [20, Satz 5, p. 619] *Let V be a finite dimensional vector space over the field $K = GF(2)$ and let $f : V \rightarrow V$ be nilpotent. The following statements are equivalent.*

- (i) *There exists a characteristic subspace of V that is not hyperinvariant.*
- (ii) *The map f has unrepeated elementary divisors λ^R and λ^S such that $R + 1 < S$.*

Suppose $X \in \text{Inv}(V, f)$. Let X_H be the largest element in $\text{Hinv}(V, f)$ such that $X_H \subseteq X$, and let X^h be the smallest element in $\text{Hinv}(V, f)$ such that

$X \subseteq X^h$. Then $X_H \subseteq X \subseteq X^h$. We call X_H and X^h the *hyperinvariant kernel* and the *hyperinvariant hull* of X , respectively, and we say that the pair (X_H, X^h) is the *hyperinvariant frame* of X . Thus, $X \in \text{Chinv}(V, f)$ is not hyperinvariant if and only if $X_H \subsetneq X \subsetneq X^h$. In this paper we study pairs (X_H, X^h) which can occur as hyperinvariant frames of characteristic non-hyperinvariant subspaces. We shall see that all elements of corresponding intervals $[X_H, X^h]$ are characteristic subspaces. We regard this fact as essential for further investigations of the lattice structure of $\text{Chinv}(V, f)$. Our main results are contained in Sections 2 - 4. In Section 1.1 we introduce basic concepts such as exponent and height, generator tuples and the group of f -commuting automorphisms. Related auxiliary material is gathered together in Section 1.3. Hyperinvariant subspaces are reviewed in Section 1.2.

We remark that Shoda [20] deals with abelian groups. But it is known (see e.g. [7]) that in many instances methods or concepts of abelian group theory can be applied to linear algebra if they are translated to modules over principal ideal domains and then specialized to $K[\lambda]$ -modules. On the other hand there are parts of linear algebra that can be interpreted in the framework of abelian group theory. In our case the language would change, and proofs would carry over almost verbatim to finite abelian p -groups [11]. Instead of hyperinvariant subspaces one would deal with subgroups that are fully invariant, and instead of characteristic non-hyperinvariant subspaces with irregular characteristic subgroups [5], [14].

1.1 Notation and basic concepts

Let $x \in V$. Define $f^0 x = x$. The smallest nonnegative integer ℓ with $f^\ell x = 0$ is called the *exponent* of x . We write $e(x) = \ell$. A nonzero vector x is said to have *height* q if $x \in f^q V$ and $x \notin f^{q+1} V$. In this case we write $h(x) = q$. We set $h(0) = \infty$. The group of automorphisms of V that commute with f will be denoted by $\text{Aut}(V, f)$. Then $\text{Aut}(V, f) \subseteq \text{End}(V, f)$, where $\text{End}(V, f)$ is the algebra of all endomorphisms of V that commute with f . Clearly, if $\alpha \in \text{Aut}(V, f)$ then $\alpha(f^i x) = f^i(\alpha x)$ for all $x \in V$. Hence $e(\alpha x) = e(x)$ and $h(\alpha x) = h(x)$ for all $x \in V, \alpha \in \text{Aut}(V, f)$. We set $V[f^j] = \text{Ker } f^j, j \geq 0$. Thus, the assumption $f^n = 0$ implies $V = V[f^n]$. Let

$$\begin{aligned} \langle x \rangle &= \text{span}\{f^i x, i \geq 0\} = \\ &\quad \{c_0 x + c_1 f x + \cdots + c_{n-1} f^{n-1} x; c_i \in K, i = 0, 1, \dots, n-1\} \end{aligned}$$

be the f -cyclic subspace generated by x . To $B \subseteq V$ we associate the subspaces $\langle B \rangle = \sum_{b \in B} \langle b \rangle$, and

$$B^c = \cap \{W \in \text{Chinv}(V, f); B \subseteq W\}, \quad B^h = \cap \{W \in \text{Hinv}(V, f); B \subseteq W\}.$$

Then

$$\langle B \rangle^c = \langle \alpha b; b \in B, \alpha \in \text{Aut}(V, f) \rangle = \sum_{b \in B} \langle b \rangle^c,$$

and

$$\langle B \rangle^h = \langle \eta b; b \in B, \eta \in \text{End}(V, f) \rangle = \sum_{b \in B} \langle b \rangle^h,$$

and $\langle B \rangle^c \subseteq \langle B \rangle^h$. We call the subspaces B^c and B^h the *characteristic hull* and the *hyperinvariant hull* of B , respectively. A subspace X is hyperinvariant if and only if $X = \langle X \rangle^c = \langle X \rangle^h$.

Suppose $\dim \text{Ker } f = m$. Let $\lambda^{t_1}, \dots, \lambda^{t_m}$ be the elementary divisors of f such that $t_1 + \dots + t_m = \dim V$. Then V can be decomposed into a direct sum of f -cyclic subspaces $\langle u_j \rangle$ such that

$$V = \langle u_1 \rangle \oplus \dots \oplus \langle u_m \rangle \quad \text{and} \quad e(u_j) = t_j, \quad j = 1, \dots, m. \quad (1.1)$$

Let $\pi_j : V \rightarrow V$, $j = 1, \dots, m$, be projections be defined by

$$\text{Im } \pi_j = \langle u_j \rangle \quad \text{and} \quad \text{Ker } \pi_j = \langle u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_m \rangle.$$

Note that $\pi_j \in \text{End}(V, f)$. If (1.1) holds and

$$0 < t_1 \leq \dots \leq t_m, \quad (1.2)$$

then we say that $U = (u_1, \dots, u_m)$ is a *generator tuple* of V (with respect to f). The tuple (t_m, \dots, t_1) of exponents - written in nonincreasing order - is known as *Segre characteristic* of f . The set of generator tuples of V will be denoted by \mathcal{U} . We call $u \in V$ a *generator* of V (see also [10, p.4]) if $u \in U$ for some $U \in \mathcal{U}$. Then $u \in V$ is a generator if and only if $u \neq 0$ and

$$V = \langle u \rangle \oplus V_2 \quad \text{for some} \quad V_2 \in \text{Inv}(V, f).$$

Unrepeated elementary divisors λ^{t_i} and corresponding generators will play a crucial role in this paper. Therefore we single out the corresponding unrepeated exponents t_i and define a set of indices

$$I_u = \{i; t_i \neq t_k \text{ if } k \neq i, 1 \leq k \leq m\}.$$

Hence we have $i \in I_u$ if and only if

$$\dim (V[f] \cap f^{t_i-1}V / V[f] \cap f^{t_i}V) = 1. \quad (1.3)$$

The left-hand side of (1.3) is the $(t_i - 1)$ -th Ulm invariant of f (see [10, p.154], [13, p.27]). We say that a generator u is unrepeated if $e(u) = t_i$ and $i \in I_u$.

1.2 Hyperinvariant subspaces

Let $U = (u_1, \dots, u_m) \in \mathcal{U}$ be a generator tuple such that (1.1) and (1.2) hold. Define $\vec{t} = (t_1, \dots, t_m)$. Let $\mathcal{L}(\vec{t})$ be the set of m -tuples $\vec{r} = (r_1, \dots, r_m)$ of integers satisfying

$$0 \leq r_1 \leq \dots \leq r_m \text{ and } 0 \leq t_1 - r_1 \leq \dots \leq t_m - r_m. \quad (1.4)$$

We write $\vec{r} \preceq \vec{s}$ if $\vec{r} = (r_j)_{j=1}^m$, $\vec{s} = (s_j)_{j=1}^m \in \mathcal{L}(\vec{t})$ and $r_j \leq s_j$, $1 \leq j \leq m$. Then $(\mathcal{L}(\vec{t}), \preceq)$ is a lattice. The following theorem is due to Fillmore, Herrero and Longstaff [9]. We refer to [12] for a proof. A related result concerning fully invariant subgroups of abelian p -groups is Theorem 2.8 in [8].

Theorem 1.2. *Let $f : V \rightarrow V$ be nilpotent.*

(i) *If $\vec{r} \in \mathcal{L}(\vec{t})$, then*

$$W(\vec{r}) = f^{r_1}V \cap V[f^{t_1-r_1}] + \dots + f^{r_m}V \cap V[f^{t_m-r_m}]$$

is a hyperinvariant subspace. Conversely, each $W \in \text{Hinv}(V, f)$ is of the form $W = W(\vec{r})$ for some $\vec{r} \in \mathcal{L}(\vec{t})$.

(ii) *If $\vec{r} \in \mathcal{L}(\vec{t})$ then $W(\vec{r}) = f^{r_1}\langle u_1 \rangle \oplus \dots \oplus f^{r_m}\langle u_m \rangle$.*

(iii) *The mapping $\vec{r} \mapsto W(\vec{r})$ is a lattice isomorphism from $(\mathcal{L}(\vec{t}), \preceq)$ onto $(\text{Hinv}(V, f), \supseteq)$.*

Let $X \in \text{Chinv}(V, f)$. The first part of Theorem 1.3 below deals with the hyperinvariant kernel X_H of X . In [18] the theorem is used to obtain a description of the set $\text{Chinv}(V, f) \setminus \text{Hinv}(V, f)$.

Theorem 1.3. [2] *Suppose X is a characteristic subspace of V . Let $U = (u_1, \dots, u_m) \in \mathcal{U}$.*

(i) *Then $X_H = \bigoplus_{j=1}^m (X \cap \langle u_j \rangle)$.*

(ii) *The subspace X is hyperinvariant if and only if*

$$\pi_j X = X \cap \langle u_j \rangle \text{ for all } j \in \{1, \dots, m\}. \quad (1.5)$$

(iii) *If $j \notin I_u$ then $\pi_j X = X \cap \langle u_j \rangle$. If $|I_u| \leq 1$, that is, if f has at most one unrepeated elementary divisor, then X is hyperinvariant.*

The following observation is related to (1.5).

Lemma 1.4. *Let $U = (u_j)_{j=1}^m \in \mathcal{U}$. Suppose $X \in \text{Inv}(V, f)$. Then*

$$X \cap \langle u_j \rangle = \langle f^{r_j} u_j \rangle \text{ and } \pi_j X = \langle f^{\mu_j} u_j \rangle \text{ with } 0 \leq \mu_j \leq r_j \leq t_j. \quad (1.6)$$

Proof. We have $X \cap \langle u_j \rangle \in \text{Inv}(V, f)$, and because of $\pi_j f = f \pi_j$ we also have $\pi_j X \in \text{Inv}(V, f)$. The invariant subspaces of $\langle u_j \rangle$ are $\langle f^s u_j \rangle$, $s = 0, \dots, t_j$. Hence $X \cap \langle u_j \rangle \subseteq \pi_j X$ yields (1.6). \square

Let $X \in \text{Chinv}(V, f)$. Theorem 2.1 will show that the numbers r_j and μ_j in (1.6) satisfy $r_j - \mu_j \leq 1$. We shall see that generators $u_j \in U$ with $\langle u_j \rangle \cap X \subsetneq \pi_j X$ require special attention. For that reason we associate to X the set

$$J(X) = \{j; \langle u_j \rangle \cap X \subsetneq \pi_j X\}.$$

We see from Theorem 1.3(ii) that X is hyperinvariant if and only if $J(X)$ is empty. Moreover, Theorem 1.3(iii) implies $J(X) \subseteq I_u$.

1.3 Generators and images under automorphisms

In this section we derive an auxiliary result which we shall use to determine the characteristic hull of subsets B of V . Let $U = (u_1, \dots, u_m) \in \mathcal{U}$ and $\alpha \in \text{Aut}(V, f)$. Then $\alpha U \in \mathcal{U}$. On the other hand, if $U' = (u'_1, \dots, u'_m) \in \mathcal{U}$ then a mapping $\alpha : U \rightarrow U'$, $\alpha : u_j \mapsto u'_j$, $j = 1, \dots, m$, can be extended to a unique $\alpha \in \text{Aut}(V, f)$. We first note an equivalent characterization of generators.

Lemma 1.5. [3, Lemma 2.6] *Suppose λ^t is an elementary divisor of f . Then x is a generator of V with $e(x) = t$ if and only if $f^t x = 0$ and*

$$h(x) = 0 \quad \text{and} \quad h(f^{t-1}x) = t - 1. \quad (1.7)$$

The condition (1.7) is equivalent to $h(f^r x) = r$, $r = 0, 1, \dots, t - 1$.

Let $u_i \in U = (u_1, \dots, u_m)$ be an unrepeated generator. If $x \in V$ is a generator with $e(x) = e(u_i)$ then $U' = (u_1, \dots, u_{i-1}, x, u_{i+1}, \dots, u_m) \in \mathcal{U}$. We say that the corresponding f -automorphism $\alpha : U \rightarrow U'$ exchanges u_i by x , and we denote it by $\alpha(u_i, x)$. The next lemma describes the elements x that we can choose for the replacement of an unrepeated generator. Let $[x] = \{\alpha x; \alpha \in \text{Aut}(V, f)\}$ denote the orbit of $x \in V$ under $\text{Aut}(V, f)$.

Lemma 1.6. *Let $U = (u_1, \dots, u_m) \in \mathcal{U}$ and suppose $u_i \in U$ is unrepeated and $e(u_i) = t_i = t$. Then x is a generator of V with $e(x) = t$ if and only if*

$$x = u_i + v + y \quad \text{with} \quad v \in \langle f u_i \rangle \quad \text{and} \quad y \in \sum_{j=1, j \neq i}^m \langle u_j \rangle [f^t]. \quad (1.8)$$

Moreover

$$[u_i] = u_i + \langle fu_i \rangle + \sum_{j=1, j \neq i}^m \langle u_j \rangle [f^t]. \quad (1.9)$$

Proof. If (1.8) holds then x satisfies $e(x) = t$ and (1.7). Hence x is a generator. Set $L^{(i)} = \langle u_1, \dots, u_{i-1} \rangle$ and $G^{(i)} = \langle u_{i+1}, \dots, u_m \rangle$. Then

$$V = L^{(i)} \oplus \langle u_i \rangle \oplus G^{(i)}, \quad (1.10)$$

and $V[f^t] = L^{(i)} \oplus \langle u_i \rangle \oplus G^{(i)}[f^t]$. Let $x = x_L + x_i + x_G \in V$ be decomposed in accordance with (1.10). If $f^t x = 0$ then $x_G \in G^{(i)}[f^t]$. Moreover, if $x_G \neq 0$, then $\langle u_j \rangle [f^t] = \langle f^{t_j-t} u_j \rangle$ and $e(u_j) > t$ yield $h(x_G) \geq 1$. Now suppose $e(x) = t$ and (1.7). Then $h(x) = 0$ and $h(x_G) \geq 1$ imply $h(x_L + x_i) = 0$. From $f^{t-1} x_L = 0$ and $h(f^{t-1} x) = t - 1$ we obtain $h(x_i) = 0$, that is, $x_i = u_i + v$, $v \in \langle fu_i \rangle$. It follows from (1.8) that $[u_i]$ is a linear manifold of the form (1.9). \square

In the course of our paper we shall frequently illustrate our results by a running example. For that purpose we always use a vector space V of dimension 10 and an endomorphism f of V with elementary divisors $\lambda, \lambda^3, \lambda^6$ such that

$$V = \langle u_1 \rangle \oplus \langle u_2 \rangle \oplus \langle u_3 \rangle \quad \text{and} \quad (e(u_1), e(u_2), e(u_3)) = (1, 3, 6). \quad (1.11)$$

In the following example we apply Lemma 1.6 to determine the characteristic hull of subspaces.

Example 1.7. Let (V, f) be given by (1.11). We consider two subspaces, namely

$$G = \langle z \rangle^c \quad \text{with} \quad z = u_1 + fu_2 + f^2u_3, \quad (1.12)$$

and

$$F = \langle w_1, w_2 \rangle^c \quad \text{with} \quad w_1 = u_1 + fu_2, \quad w_2 = fu_2 + f^2u_3. \quad (1.13)$$

We have

$$\begin{aligned} [u_1] &= u_1 + \langle u_2 \rangle [f] + \langle u_3 \rangle [f] = u_1 + \langle f^2u_2 \rangle + \langle f^5u_3 \rangle, \\ [u_2] &= u_2 + \langle fu_2 \rangle + \langle u_1 \rangle + \langle u_3 \rangle [f^3] = u_2 + \langle fu_2 \rangle + \langle u_1 \rangle + \langle f^3u_3 \rangle, \\ [u_3] &= u_3 + \langle fu_3 \rangle + \langle u_1 \rangle + \langle u_2 \rangle, \end{aligned} \quad (1.14)$$

and

$$\begin{aligned} [fu_2] &= fu_2 + \langle f^2u_2 \rangle + \langle f^4u_3 \rangle, \\ [f^2u_3] &= f^2u_3 + \langle f^3u_3 \rangle + \langle f^2u_2 \rangle. \end{aligned} \quad (1.15)$$

Hence $[z] = z + \langle f^2u_2, f^3u_3 \rangle$ and

$$G = \langle z \rangle^c = \langle z, f^2u_2, f^3u_3 \rangle.$$

From (1.14) and (1.15) we obtain

$$[w_1] = u_1 + \langle f^2u_2 \rangle + \langle f^5u_3 \rangle + fu_2 + \langle f^2u_2 \rangle + \langle f^4u_3 \rangle = w_1 + \langle f^2u_2, f^4u_3 \rangle$$

and

$$\begin{aligned} [w_2] = fu_2 + \langle f^2u_2 \rangle + \langle f^4u_3 \rangle + f^2u_3 + \langle f^3u_3 \rangle + \langle f^2u_2 \rangle = \\ w_2 + \langle f^2u_2 \rangle + \langle f^3u_3 \rangle, \end{aligned}$$

$$F = \langle w_1, w_2 \rangle^c = \langle w_1, w_2, f^2u_2, f^3u_3 \rangle = \langle w_1, w_2 \rangle.$$

Let $Q \in \{G, F\}$. Then $Q \cap \langle u_1 \rangle = 0$, $Q \cap \langle u_2 \rangle = \langle f^2u_2 \rangle$, and $Q \cap \langle u_3 \rangle = \langle f^3u_3 \rangle$. Thus

$$G_H = F_H = \langle f^2u_2, f^3u_3 \rangle = W(1, 2, 3). \quad (1.16)$$

We have $\pi_1 z = u_1 \notin G$. Therefore $\pi_1 \in \text{End}(V, f)$ implies that the characteristic subspace G is not hyperinvariant. Similarly we conclude from $\pi_1 w_1 = u_1 \notin F$ that F is not hyperinvariant.

2 Frames

In this section we consider the hyperinvariant frame (X_H, X^h) of a characteristic subspace X and we describe the connection between X_H and X^h .

Theorem 2.1. *Let X be a characteristic subspace of V and let the numbers r_j, μ_j , $j = 1, \dots, m$, be given by*

$$\langle u_j \rangle \cap X = \langle f^{r_j}u_j \rangle \text{ and } \pi_j X = \langle f^{\mu_j}u_j \rangle, \quad j = 1, \dots, m. \quad (2.1)$$

(i) *Then*

$$\mu_j = \begin{cases} r_j & \text{if } \langle u_j \rangle \cap X = \pi_j X \\ r_j - 1 & \text{if } \langle u_j \rangle \cap X \subsetneq \pi_j X. \end{cases} \quad (2.2)$$

(ii) *The hyperinvariant hull of X is*

$$X^h = \sum_{j=1}^m \pi_j X = \langle f^{\mu_1}u_1, \dots, f^{\mu_m}u_m \rangle. \quad (2.3)$$

Proof. (i) If $X \cap \langle u_j \rangle = \pi_j X$ then $\mu_j = r_j$. Now suppose $X \cap \langle u_j \rangle \subsetneq \pi_j X$. Then $\langle f^{r_j} u_j \rangle \subsetneq \langle f^{\mu_j} u_j \rangle$ implies $r_j > \mu_j$, and therefore $r_j \geq 1$. Because of $f^{\mu_j} u_j \in \pi_j X$ we can choose an element $x \in X$ such that

$$x = \sum_{i=1}^m x_i, \quad x_i \in \langle u_i \rangle, \quad i = 1, \dots, m, \quad \text{and} \quad x_j = \pi_j x = f^{\mu_j} u_j.$$

Let $\alpha = \alpha(u_j, u_j + f u_j)$. Then $\alpha x = x + f^{\mu_j+1} u_j$. By assumption the subspace X is characteristic. Therefore $\alpha x \in X$. Hence $f^{\mu_j+1} u_j \in X \cap \langle u_j \rangle = \langle f^{r_j} u_j \rangle$. This implies $\mu_j + 1 \geq r_j$, and yields $\mu_j = r_j - 1$, and completes the proof of (2.2).

(ii) Set $\tilde{X} = \sum_{j=1}^m \pi_j X$. Then $X \subseteq \tilde{X}$ and $\pi_j \tilde{X} = \pi_j X$, $j = 1, \dots, m$. Hence

$$\tilde{X} = \sum_{j=1}^m \pi_j \tilde{X} = \sum_{j=1}^m \langle f^{\mu_j} u_j \rangle. \quad (2.4)$$

Let us show that \tilde{X} is hyperinvariant. We first prove that \tilde{X} is characteristic. We consider the generators $f^{\mu_j} u_j$ of \tilde{X} . Let $\alpha \in \text{Aut}(V, f)$. If $\langle u_j \rangle \cap X = \pi_j X$ then

$$\langle f^{\mu_j} u_j \rangle = \langle f^{r_j} u_j \rangle = \langle u_j \rangle \cap X \subseteq X,$$

and therefore $\alpha(f^{\mu_j} u_j) \in X \subseteq \tilde{X}$. If $\langle u_j \rangle \cap X \subsetneq \pi_j X$, then u_j is unrepeated. Hence $\alpha u_j = u_j + y$ with $y \in \langle u_1, \dots, u_{j-1}, f u_j, u_{j+1}, \dots, u_m \rangle$, and $u_j + y$ is a generator with $e(u_j + y) = e(u_j)$. Then

$$\alpha(f^{\mu_j} u_j) = f^{\mu_j} u_j + f^{\mu_j} y. \quad (2.5)$$

Let us show that $f^{\mu_j} y \in \tilde{X}$. From (2.4) follows $f^{\mu_j} u_j \in \pi_j \tilde{X} = \pi_j X$. Hence $f^{\mu_j} u_j = \pi_j x$ for some $x \in X$. Then $x = x_1 + \dots + x_m$ with $x_i \in \langle u_i \rangle$, $i = 1, \dots, m$, and $x_j = f^{\mu_j} u_j$. Let $\beta \in \text{Aut}(V, f)$ be the automorphism that exchanges u_j by $u_j + y$. Then $\beta x = x + f^{\mu_j} y \in X$, which implies $f^{\mu_j} y \in X \subseteq \tilde{X}$. We have $f^{\mu_j} u_j \in \tilde{X}$. Hence (2.5) yields $\alpha(f^{\mu_j} u_j) \in \tilde{X}$. Thus we have shown that \tilde{X} is characteristic. It follows from Theorem 1.3(ii) that \tilde{X} is hyperinvariant. Then $X \subseteq \tilde{X}$ implies $X^h \subseteq \tilde{X}^h = \tilde{X}$. On the other hand we obtain $\tilde{X} \subseteq X^h$, since X^h is the hyperinvariant hull of X . Therefore $\tilde{X} = X^h$. \square

Let $\vec{e}_1 = (1, 0, \dots, 0)$, \dots , $\vec{e}_m = (0, \dots, 0, 1)$ be the row vectors of size m . We combine the preceding theorem with Theorem 1.2 and Theorem 1.3.

Corollary 2.2. *Let X be a characteristic subspace of V and let the integers r_j , μ_j , $j = 1, \dots, m$ be defined by (2.1). Set $\vec{r} = (r_1, \dots, r_m)$ and $\vec{\mu} = (\mu_1, \dots, \mu_m)$.*

(i) The hyperinvariant frame (X_H, X^h) of X consists of

$$X_H = \sum_{j=1}^m (\langle u_j \rangle \cap X) = \langle f^{r_1} u_1, \dots, f^{r_m} u_m \rangle = W(\vec{r}), \quad (2.6)$$

and

$$X^h = \sum_{j=1}^m \pi_j X = \langle f^{\mu_1} u_1, \dots, f^{\mu_m} u_m \rangle = W(\vec{\mu}).$$

If $J(X) = \{i_1, \dots, i_k\} \subset I_u$, $i_1 < \dots < i_k$, then

$$\vec{r} = \vec{\mu} + \sum_{s=1}^k \vec{e}_{i_s} \quad \text{and} \quad \vec{r}, \vec{\mu} \in \mathcal{L}(\vec{t}). \quad (2.7)$$

(ii) Let $D(X) = \text{span}\{f^{\mu_i} u_i; i \in J(X)\}$. Then $X^h = X_H \oplus D(X)$. The subspace X is hyperinvariant if and only if $D(X) = 0$.

We extend Example 1.7 taking into account the results of the the preceding corollary.

Example 2.3. Let (V, f) be given by (1.11). Let the subspaces G and F be defined by (1.12) and (1.13), respectively. Then

$$G = \langle u_1 + f u_2 + f^2 u_3, f^2 u_2, f^3 u_3 \rangle \quad \text{and} \quad F = \langle u_1 + f u_2, f u_2 + f^2 u_3 \rangle$$

yield

$$G^h = F^h = \langle u_1, f u_2, f^2 u_3 \rangle = W(\vec{\mu}) \quad \text{with} \quad \vec{\mu} = (0, 1, 2)$$

and $\dim W(\vec{\mu}) = 7$. Recall (1.16), that is

$$G_H = F_H = \langle f^2 u_2, f^3 u_3 \rangle = W(\vec{r}) \quad \text{with} \quad \vec{r} = (1, 2, 3),$$

and $\dim W(\vec{r}) = 4$. Hence $(W(1, 2, 3), W(0, 1, 2))$ is the hyperinvariant frame for both G and F . If $Q \in \{G, F\}$ then $\langle u_j \rangle \cap Q \subsetneq \pi_j Q$, $j = 1, 2, 3$, implies $J(Q) = \{1, 2, 3\}$. Then

$$\sum_{\kappa \in J(Q)} \vec{e}_\kappa = (1, 1, 1),$$

and the relation $\vec{r} = \vec{\mu} + \sum_{\kappa \in J(Q)} \vec{e}_\kappa$ in (2.7) is satisfied. Moreover, $D(Q) = \text{span}\{u_1, f u_2, f^2 u_3\}$ such that $Q^h = Q_H \oplus D$.

When is a pair $(W(\vec{r}), W(\vec{\mu}))$ the hyperinvariant frame of a subspace $X \in \text{Chinv}(V) \setminus \text{Hinv}(V, f)$? From Lemma 2.4 below we obtain necessary conditions that involve the set $J(X) = \{j; \langle u_j \rangle \cap X \subsetneq \pi_j X\}$. The strict inequality (2.10) below can be interpreted in the view of Shoda's theorem. It follows from (2.10) that a given f can give rise to a characteristic subspace that is not hyperinvariant only if f has unrepeated elementary divisors λ^R and λ^S such that the integers R and S are not consecutive.

Lemma 2.4. *Suppose X is characteristic and not hyperinvariant with*

$$X^h = W(\mu_1, \dots, \mu_m) \quad \text{and} \quad X_H = W(r_1, \dots, r_m).$$

Then $|J(X)| \geq 2$ and $\vec{\mu}$ has the following properties.

(i) *If $p \in J(X)$ then*

$$\mu_p < \mu_q \quad \text{if } p < q. \quad (2.8)$$

(ii) *If $q \in J(X)$ then $0 < t_q - \mu_q$ and*

$$t_p - \mu_p < t_q - \mu_q \quad \text{if } p < q. \quad (2.9)$$

(iii) *If $p, q \in J(X)$ then*

$$t_p + 1 < t_q \quad \text{if } p < q. \quad (2.10)$$

Proof. We show that $|J(X)| > 1$. Suppose X is characteristic and

$$\langle u_j \rangle \cap X = \pi_j X \quad \text{for all } j \in \{1, \dots, m\} \setminus \{s\}. \quad (2.11)$$

Let $x \in X$ be written as $x = \sum_{j=1}^m x_j$, $x_j = \pi_j x$, $j = 1, \dots, m$. Then (2.11) implies $x_s \in X$. Hence $\langle u_s \rangle \cap X = \pi_s X$. Therefore X is hyperinvariant (by Theorem 1.3).

(i) Suppose $p \in J(X)$ and $p < q$. Since u_p is unrepeated we have $t_p < t_q$. From $e(u_p) < e(u_q)$ follows $e(u_q + u_p) = e(u_q)$. Let $\alpha = \alpha(u_q, u_q + u_p)$. Then $\alpha X \subseteq X$. From

$$f^{\mu_q} u_q \in W(\vec{\mu}) = X^h = \sum_{j=1}^m \pi_j X$$

follows $f^{\mu_q} u_q \in \pi_q X$. Therefore $f^{\mu_q} u_q = \pi_q x$ for some $x \in X$. Then $\alpha x = x + f^{\mu_q} u_p \in X$, and therefore $f^{\mu_q} u_p \in X$. Hence

$$f^{\mu_q} u_p \in X \cap \langle u_p \rangle = \langle f^{r_p} u_p \rangle = \langle f^{\mu_p+1} u_p \rangle,$$

which implies $\mu_q \geq \mu_p + 1$.

(ii) If $0 = t_i - \mu_i$ then $\pi_i X = \langle f^{\mu_i} u_i \rangle = 0$. Hence $0 = \langle u_i \rangle \cap X = \pi_i X$, and therefore $i \notin J(X)$. Suppose $q \in J(X)$ and $p < q$. Then $t_p < t_q$, and therefore

$$e(u_p + f^{t_q - t_p} u_q) = e(u_p) = t_p.$$

Let $\alpha = \alpha(u_p, u_p + f^{t_q - t_p} u_q)$. Because of $f^{\mu_p} u_p \in \pi_p X$ there exists an $x \in X$ such that $\pi_p x = f^{\mu_p} u_p$. Then $\alpha x = x + f^{\mu_p + t_q - t_p} u_q \in X$. Hence

$$f^{\mu_p + t_q - t_p} u_q \in X \cap \langle u_q \rangle = \langle f^{r_q} u_q \rangle = \langle f^{\mu_q+1} u_q \rangle.$$

Therefore $\mu_p + t_q - t_p \geq \mu_q + 1$, which implies $t_q - \mu_q > t_p - \mu_p$.

(iii) Suppose $p, q \in J(X)$, $p < q$. Then (2.8) and (2.9) imply $1 \leq \mu_q - \mu_p < t_q - t_p$. Hence $t_p + 1 < t_q$. \square

From Theorem 2.1 and Lemma 2.4 we obtain the following.

Theorem 2.5. *Let $\vec{\mu}, \vec{r} \in \mathcal{L}(\vec{t})$ and let*

$$J = \{i_1, \dots, i_k\} \subset I_u, \quad i_1 < \dots < i_k, \quad |J| \geq 2. \quad (2.12)$$

If X is characteristic and not hyperinvariant and $J(X) = J$ and $X^h = W(\vec{\mu})$ and $X_H = W(\vec{r})$, then

$$\vec{r} = \vec{\mu} + \sum_{s=1}^k \vec{e}_{i_s},$$

and $\vec{\mu}_J = (\mu_{i_1}, \dots, \mu_{i_k})$ satisfies

$$0 \leq \mu_{i_1} < \dots < \mu_{i_k} \quad \text{and} \quad 0 < t_{i_1} - \mu_{i_1} < \dots < t_{i_k} - \mu_{i_k}, \quad (2.13)$$

and

$$t_{i_s} + 1 < t_{i_{(s+1)}}, \quad s = 1, \dots, k-1. \quad (2.14)$$

Assuming (2.12) we prove in Section 4 a converse of the preceding theorem. If the entries of $\vec{t}_J = (t_{i_1}, \dots, t_{i_k})$ satisfy (2.14) then there exist tuples $\vec{\mu}_J = (\mu_{i_1}, \dots, \mu_{i_k})$ of nonnegative integers such that the inequalities (2.13) hold. One can check that $\vec{\mu}_J$ satisfies (2.13) if and only if $0 \leq \mu_{i_1} < t_{i_1}$ and $\mu_{i_{s+1}} = \mu_{i_s} + \delta_s$ with $1 \leq \delta_s < t_{i_{s+1}} - t_{i_s}$, $s = 1, \dots, k-1$. In Lemma 4.1 we shall see that one can extend such a $\vec{\mu}_J$ to an m -tuple $\vec{\mu}$ such that $\vec{\mu} \in \mathcal{L}(\vec{t})$ and $\vec{\mu} + \sum_{j \in J} \vec{e}_j \in \mathcal{L}(\vec{t})$. Then, using Theorem 3.3 one can construct a characteristic non-hyperinvariant subspace X such that $J(X) = J$, and

$$X_H \cap \langle u_j \rangle = \langle f^{\mu_j+1} u_j \rangle \quad \text{and} \quad X^h \cap \langle u_j \rangle = \langle f^{\mu_j} u_j \rangle, \quad j \in J.$$

3 Intervals

Let $A, B \in \text{Inv}(V, f)$ and $A \subseteq B$. The set

$$[A, B] = \{C \in \text{Inv}(V, f), \quad A \subseteq C \subseteq B\}$$

is an interval of the invariant subspace lattice $\text{Inv}(V, f)$. In this section we study intervals of the form $[X_H, X^h]$, which can arise from subspaces $X \in \text{Chinv}(V, f) \setminus \text{Hinv}(V, f)$. A useful property of direct sums and intervals is the following.

Lemma 3.1. [10, p. 38] *Let A, B, C, D be subspaces of V . Suppose $B = A \oplus D$ and $C \in [A, B]$. Then $Z = C \cap D$ is the unique subspace satisfying*

$$Z \subseteq D \quad \text{and} \quad C = A \oplus Z. \quad (3.1)$$

Proof. The modular law implies $C = B \cap C = (A \oplus D) \cap C = A \oplus (D \cap C)$. Hence $Z = D \cap C$ has the properties (3.1). Conversely, if (3.1) holds, then $C \cap D = (A \oplus Z) \cap D = (A \cap D) \oplus Z = Z$. \square

For the proof of Theorem 3.3 we need the following auxiliary result.

Lemma 3.2. *Let $J = \{i_1, \dots, i_k\} \subseteq I_u$, $i_1 < \dots < i_k$, $2 \leq k$. Suppose $\vec{\mu} = (\mu_1, \dots, \mu_m) \in \mathcal{L}(\vec{t})$ and*

$$0 \leq \mu_{i_1} < \mu_{i_2} < \dots < \mu_{i_k} \quad (3.2)$$

and

$$0 < t_{i_1} - \mu_{i_1} < t_{i_2} - \mu_{i_2} < \dots < t_{i_k} - \mu_{i_k}, \quad (3.3)$$

and suppose $\vec{r} = \vec{\mu} + \sum_{s=1}^k \vec{e}_{i_s} \in \mathcal{L}(\vec{t})$. Let $U = (u_1, \dots, u_m) \in \mathcal{U}$ and $\alpha \in \text{Aut}(V, f)$.

(i) Then

$$\alpha f^{\mu_{i_s}} u_{i_s} = f^{\mu_{i_s}} u_{i_s} + w_{i_s} \quad \text{with} \quad w_{i_s} \in W(\vec{r}), \quad s = 1, \dots, k. \quad (3.4)$$

(ii) If $z \in \text{span}\{f^{\mu_{i_1}} u_{i_1}, \dots, f^{\mu_{i_k}} u_{i_k}\}$ then $\alpha z = z + w$ with $w \in W(\vec{r})$.

Proof. (i) The generator u_{i_s} is unrepeatd. Therefore Lemma 1.6 yields

$$\begin{aligned} \alpha u_{i_s} &= u_{i_s} + v_{i_s} + y_{i_s} \quad \text{with} \quad v_{i_s} \in \langle f u_{i_s} \rangle, \quad \text{and} \\ y_{i_s} &\in \langle u_j; j = 1, \dots, m; j \neq i_s \rangle, \quad e(y_{i_s}) \leq e(u_{i_s}) = t_{i_s}, \quad s = 1, \dots, k. \end{aligned}$$

Then

$$\alpha f^{\mu_{i_s}} u_{i_s} = f^{\mu_{i_s}} u_{i_s} + w_{i_s} \quad \text{with} \quad w_{i_s} = f^{\mu_{i_s}} v_{i_s} + f^{\mu_{i_s}} y_{i_s}.$$

We have $\alpha f^{\mu_{i_s}} u_{i_s} \in W(\vec{\mu})$, since $f^{\mu_{i_s}} u_{i_s} \in W(\vec{\mu})$ and $W(\vec{\mu})$ is hyperinvariant. Moreover $f^{\mu_{i_s}} v_{i_s} \in \langle f^{\mu_{i_s}+1} u_{i_s} \rangle \subseteq W(\vec{r}) \subseteq W(\vec{\mu})$. Hence

$$f^{\mu_{i_s}} y_{i_s} \in W(\vec{\mu}) = \langle f^{\mu_1} u_1, \dots, f^{\mu_m} u_m \rangle. \quad (3.5)$$

It remains to show that $f^{\mu_{i_s}} y_{i_s} \in W(\vec{r})$. Let y_{i_s} be written as

$$y_{i_s} = \sum_{j=1, j \neq i_s}^m x_j \quad \text{with} \quad x_j \in \langle u_j \rangle.$$

Then (3.5) implies

$$f^{\mu_{i_s}} x_j \in W(\vec{\mu}) \cap \langle u_j \rangle = \langle f^{\mu_j} u_j \rangle. \quad (3.6)$$

If $j \notin J = \{i_1, \dots, i_k\}$ then $r_j = \mu_j$, and (3.6) yields $f^{\mu_{i_s}} x_j \in W(\vec{r})$. Suppose $j \in J$ and $j > i_s$. Then $e(x_j) \leq e(y_{i_s}) \leq t_{i_s}$ implies $x_j \in f^{t_j - t_{i_s}} \langle u_j \rangle$. Hence it follows from (3.3) that

$$f^{\mu_{i_s}} x_j \in f^{t_j - t_{i_s} + \mu_{i_s}} \langle u_j \rangle = f^{\mu_j + (t_j - \mu_j) - (t_{i_s} - \mu_{i_s})} \langle u_j \rangle \subseteq f^{\mu_j + 1} \langle u_j \rangle = f^{r_j} \langle u_j \rangle,$$

and we see in this case that $f^{\mu_{i_s}} x_j \in W(\vec{r})$. Now suppose $j \in J$ and $i_s > j$. If $j = i_\tau$, $\tau < s$ then (3.2) implies $\mu_{i_s} > \mu_{i_\tau} = \mu_j$, and we obtain

$$f^{\mu_{i_s}} x_j \in f^{\mu_{i_s}} \langle u_j \rangle \subseteq f^{\mu_j + 1} \langle u_j \rangle = f^{r_j} \langle u_j \rangle,$$

and therefore $f^{\mu_{i_s}} x_j \in W(\vec{r})$. Hence $f^{\mu_{i_s}} y_{i_s} \in W(\vec{r})$, which completes the proof of (3.4).

(ii) Let $z = \sum_{s=1}^k c_s f^{\mu_{i_s}} u_{i_s}$, $c_s \in K$. Then (3.4) implies

$$\alpha z = z + \sum_{s=1}^k c_s w_{i_s} \in z + W(\vec{r}).$$

□

We have seen in Theorem 2.5 that a subspace $X \in \text{Chinv}(V, f) \setminus \text{Hinv}(V, f)$ with $X_H = W(\vec{r})$, $X^h = W(\vec{\mu})$ and $J(X) = \{i_s\}_{s=1}^k$ satisfies the conditions (3.7) - (3.9) of Theorem 3.3 below. Hence, if (X_H, X^h) is the hyperinvariant frame of X then the following theorem describes the corresponding interval $[X_H, X^h]$.

Theorem 3.3. *Let $J = \{i_1, \dots, i_k\} \subseteq I_u$, $i_1 < \dots < i_k$, $2 \leq k$. Assume*

$$t_{i_s} + 1 < t_{i_{(s+1)}}, \quad s = 1, \dots, k-1. \quad (3.7)$$

Let $\vec{\mu}, \vec{r} \in \mathcal{L}(\vec{t})$ be such that

$$0 \leq \mu_{i_1} < \dots < \mu_{i_k} \quad \text{and} \quad 0 < t_{i_1} - \mu_{i_1} < \dots < t_{i_k} - \mu_{i_k} \quad (3.8)$$

and

$$\vec{r} = \vec{\mu} + \sum_{s=1}^k \vec{e}_{i_s} \quad (3.9)$$

hold. Set

$$D_{\vec{\mu}_J} = \text{span}\{f^{\mu_{i_1}} u_{i_1}, \dots, f^{\mu_{i_k}} u_{i_k}\}. \quad (3.10)$$

(i) *Each subspace $X \in [W(\vec{r}), W(\vec{\mu})]$ is characteristic. Moreover, $X \in [W(\vec{r}), W(\vec{\mu})]$ if and only if $X = W(\vec{r}) \oplus Z$ for some subspace $Z \subseteq D_{\vec{\mu}_J}$.*

(ii) *A subspace $X \in [W(\vec{r}), W(\vec{\mu})]$ is hyperinvariant if and only if*

$$X = W(\vec{r}) \oplus \text{span}\{f^{\mu_{\tau_1}} u_{\tau_1}, \dots, f^{\mu_{\tau_q}} u_{\tau_q}\} \quad (3.11)$$

for some subset $T = \{\tau_1, \dots, \tau_q\}$ of J .

Proof. (i) Corollary 2.2(ii) implies $W(\vec{\mu}) = W(\vec{r}) \oplus D_{\vec{\mu}, J}$. If

$$W(\vec{r}) \subseteq X \subseteq W(\vec{\mu}) \quad (3.12)$$

then the subspace $Z = X \cap D_{\vec{\mu}, J}$ satisfies $X = W(\vec{r}) \oplus Z$ (by Lemma 3.1). Let $x \in X$. Then $x = y + z$ with $y \in W(\vec{r})$, $z \in Z$. If $\alpha \in \text{Aut}(f, V)$ then Lemma 3.2 implies $\alpha z = w + z$, $w \in W(\vec{r})$. Since $W(\vec{r})$ is hyperinvariant we have $\alpha y \in W(\vec{r})$, and we obtain $\alpha x \in W(\vec{r}) \oplus Z = X$.

(ii) A subspace X is hyperinvariant and satisfies (3.12) if and only if $X = W(\vec{\eta})$ for some $\vec{\eta} \in \mathcal{L}(\vec{t})$ with $\vec{\mu} \preceq \vec{\eta} \preceq \vec{r}$, that is, if and only if $\vec{\eta} = \vec{\mu} + \sum_{\nu \in T} \vec{e}_\nu$ for some subset T of J . \square

In [18] a subspace Y is called a minext subspace if it complements a hyperinvariant subspace W such that $X = W \oplus Y$ is characteristic non-hyperinvariant and $X_H = W$.

Example 3.4. Let (V, f) be given by (1.11). Then $\vec{t} = (1, 3, 6)$ and $I_u = \{1, 2, 3\}$. The sets J with property (3.7) are $J = \{1, 2, 3\}$, $J = \{1, 2\}$, $J = \{1, 3\}$, $J = \{2, 3\}$. In the following we consider $J = \{1, 2, 3\}$ and $J = \{1, 3\}$.

Case $J = \{1, 2, 3\}$. Then $\sum_{j \in J} \vec{e}_j = (1, 1, 1)$. If $\vec{\mu} = (0, 1, 2)$, $\vec{r} = (1, 2, 3)$, or $\vec{\mu} = (0, 1, 3)$, $\vec{r} = (1, 2, 4)$ then $\vec{\mu}, \vec{r} \in \mathcal{L}(\vec{t})$ holds and (3.8) is satisfied. Let us consider in more detail the case $\vec{\mu} = (0, 1, 2)$, $\vec{r} = (1, 2, 3)$. We have

$$W(\vec{\mu}) = \langle u_1, fu_2, f^2u_3 \rangle, \quad W(\vec{r}) = \langle f^2u_2, f^3u_3 \rangle$$

and $D_{\vec{\mu}} = \text{span}\{u_1, fu_2, f^2u_3\}$. It is well known (see [15], [19]) that the number of k -dimensional subspaces of an n -dimensional vector space over the field $\text{GF}(q)$ is equal to the q -binomial coefficient

$$\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

Hence the vector space $D_{\vec{\mu}}$ has

$$\binom{3}{0}_2 + \binom{3}{1}_2 + \binom{3}{2}_2 + \binom{3}{3}_2 = 1 + 7 + 7 + 1 = 16$$

subspaces, and therefore the interval $[W(\vec{r}), W(\vec{\mu})]$ contains 16 characteristic subspaces. We have $2^3 = 8$ choices for a subset T of J . Hence there are 8 hyperinvariant subspaces in $[W(\vec{r}), W(\vec{\mu})]$, e.g.

$$W(1, 2, 3) + \langle u_1 \rangle = W(0, 2, 3) \quad \text{and} \quad W(1, 2, 3) + \langle fu_2, f^2u_3 \rangle = W(1, 1, 2). \quad (3.13)$$

Thus there are 8 subspaces in $[W(\vec{r}), W(\vec{\mu})]$ that are not hyperinvariant. Examples of such subspaces are

$$Y_2 = W(\vec{r}) \oplus \text{span}\{u_1 + fu_2\} = \langle f^2u_2, f^3u_3, u_1 + fu_2 \rangle = \langle u_1 + fu_2 \rangle^c$$

with $\dim Y_2 = 5$, and

$$Y_3 = W(\vec{r}) \oplus \text{span}\{u_1 + fu_2, f^2u_3\} = \langle u_1 + fu_2, f^2u_3 \rangle = \langle u_1 + fu_2, f^2u_3 \rangle^c$$

with $\dim Y_3 = 6$. Moreover, we know from Example 1.7 and Example 2.3 that the subspaces

$$G = \langle u_1 + fu_2 + f^2u_3 \rangle^c = W(1, 2, 3) \oplus \text{span}\{u_1 + fu_2 + f^2u_3\} \quad (3.14)$$

and

$$F = \langle u_1 + fu_2, fu_2 + f^2u_3 \rangle^c = W(1, 2, 3) \oplus \text{span}\{u_1 + fu_2, fu_2 + f^2u_3\} \quad (3.15)$$

are not hyperinvariant.

Case $J = \{1, 3\}$. There are four pairs (μ_1, μ_3) that satisfy (3.8), namely

$$(\mu_1, \mu_3) \in \{(0, 1), (0, 2), (0, 3), (0, 4)\}.$$

We focus on $(\mu_1, \mu_3) = (0, 2)$. Then $\vec{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathcal{L}(\vec{t})$ if

$$\vec{\mu} \in \{(0, 0, 2), (0, 1, 2), (0, 2, 2)\},$$

and we have $\vec{r} = \vec{\mu} + (1, 0, 1) \in \mathcal{L}(\vec{t})$ if and only if $\vec{\mu} = (0, 1, 2)$ or $\vec{\mu} = (0, 2, 2)$. Then $\vec{r} = (1, 1, 3)$ or $\vec{r} = (1, 2, 3)$, respectively, and $D_{\vec{\mu}_J} = \text{span}\{u_1, f^2u_3\}$ with $\dim D_{\vec{\mu}_J} = 2$. Hence, besides their endpoints the respective intervals $[W(\vec{r}), W(\vec{\mu})]$ contain two subspaces which are hyperinvariant, namely

$$W(\vec{r}) + \text{span}\{u_1\} = W(\vec{r} - \vec{e}_1) \quad \text{and} \quad W(\vec{r}) + \text{span}\{f^2u_3\} = W(\vec{r} - \vec{e}_3),$$

together with the non-hyperinvariant subspace

$$W(\vec{r}) + \text{span}\{z\}, \quad z = u_1 + f^2u_3.$$

In the case $(\vec{\mu}, \vec{r}) = ((0, 1, 2), (1, 1, 3))$ the elements of $[W(\vec{r}), W(\vec{\mu})]$ are $W(1, 1, 3)$, $W(0, 1, 3)$, $W(1, 1, 2)$, $W(0, 1, 2)$ and

$$W(1, 1, 3) + \text{span}\{z\} = \langle fu_2, f^3u_3, u_1 + f^2u_3 \rangle = \langle fu_2, u_1 + f^2u_3 \rangle = \langle fu_2, u_1 + f^2u_3 \rangle^c. \quad (3.16)$$

In the case $(\vec{\mu}, \vec{r}) = ((0, 2, 2), (1, 2, 3))$ the interval $[W(\vec{r}), W(\vec{\mu})]$ consists of $W(1, 2, 3)$, $W(0, 2, 3)$, $W(1, 2, 2)$, $W(0, 2, 2)$ and

$$W(1, 2, 3) + \text{span}\{z\} = \langle f^2u_2, f^3u_3, u_1 + f^2u_3 \rangle = \langle f^2u_2, u_1 + f^2u_3 \rangle = \langle u_1 + f^2u_3 \rangle^c = \langle z \rangle^c.$$

To refine Theorem 3.3 we make use of matrices in column reduced echelon form. Let $D_{\vec{\mu}_J}$ be the k -dimensional vector space in (3.10), let \mathcal{Z} denote the lattice of subspaces of $D_{\vec{\mu}_J}$ and let \mathcal{M}_c be the set of $k \times k$ matrices in column reduced echelon form. Recall that a matrix is in column reduced echelon form if it has the following properties. (i) The first non-zero entry in each column (as we go down) is a 1. (ii) These “leading 1s” occur further down as we go to the right of the matrix; (iii) In the row of a leading 1 all other entries are zero. To a matrix $M \in K^{k \times k}$ we associate the subspace

$$Z(M) = \text{span}\{z_1, \dots, z_k\} \quad \text{with} \\ (z_1, z_2, \dots, z_k) = (f^{\mu_{i_1}} u_{i_1}, f^{\mu_{i_2}} u_{i_2}, \dots, f^{\mu_{i_k}} u_{i_k})M. \quad (3.17)$$

Thus $Z \in \mathcal{Z}$ if and only if $Z = Z(M)$ for some $M \in K^{k \times k}$. If M_c is the column reduced echelon form of M then $Z(M) = Z(M_c)$. Uniqueness of M_c implies that the mapping $M_c \mapsto Z(M_c)$ is a bijection from \mathcal{M}_c onto \mathcal{Z} .

The assumptions in the following theorem are those of Theorem 3.3.

Theorem 3.5. *Let $M \in K^{k \times k}$ be in column reduced echelon form and let $Z(M)$ be the associated subspace such that $X(M) = W(\vec{r}) \oplus Z(M)$ is a characteristic subspace in $[W(\vec{r}), W(\vec{\mu})]$.*

- (i) *$X(M)$ is hyperinvariant if and only if each nonzero column of M contains exactly one entry 1.*
- (ii) *We have $X(M)_H = W(\vec{r})$ if and only if each nonzero column of M has at least two entries equal to 1.*
- (iii) *We have $X(M)^h = W(\vec{\mu})$ if and only if each row of M has at least one entry equal to 1.*

Proof. Let $i_s \in J$. Because of $Z(M) \subseteq D_{\vec{\mu}_J}$ we have either $Z(M) \cap \langle u_{i_s} \rangle = 0$ or

$$Z(M) \cap \langle u_{i_s} \rangle = \text{span}\{f^{\mu_{i_s}} u_{i_s}\}, \quad (3.18)$$

and similarly either $\pi_{i_s} Z(M) = 0$ or $\pi_{i_s} Z(M) = \text{span}\{f^{\mu_{i_s}} u_{i_s}\}$. We note that (3.18) holds if and only if the s -th column of the matrix M contains exactly one entry 1 (in row s). Moreover, $\pi_{i_s} Z(M) = 0$ holds if and only if the s -th row is the zero row. Suppose $\dim Z(M) = \text{rank } M = q$. Then $M \in \mathcal{M}_c$ implies $M = \begin{pmatrix} \tilde{M} & 0_{k \times (k-q)} \end{pmatrix}$ and $\text{rank } \tilde{M} = q$.

(i) Each nonzero column of M contains exactly one entry 1 if and only if $\Pi^{-1}M = \text{diag}(I_q, 0)$ for some permutation matrix Π . This is equivalent to

$$(f^{\mu_{i_1}} u_{i_1}, \dots, f^{\mu_{i_k}} u_{i_k})M = (f^{\mu_{i_1}} u_{i_1}, \dots, f^{\mu_{i_k}} u_{i_k})\Pi \text{diag}(I_p, 0) = \\ (f^{\mu_{\tau_1}} u_{\tau_1}, \dots, f^{\mu_{\tau_q}} u_{\tau_q}, 0, \dots, 0)$$

with $T = \{\tau_1, \dots, \tau_q\} \subseteq J = \{i_1, \dots, i_k\}$. Now we apply Theorem 3.3(ii).

(ii) In the following let $X = X(M)$. From

$$X_H = \sum_{j=1}^m (X \cap \langle u_j \rangle) = W(\vec{r}) + \sum_{s=1}^k (Z(M) \cap \langle u_{i_s} \rangle)$$

follows that $X_H = W(\vec{r})$ is equivalent to

$$Z(M) \cap \langle u_{i_s} \rangle = 0, \quad s = 1, \dots, k. \quad (3.19)$$

Condition (3.19) holds if and only if M does not contain a nonzero column with exactly one entry 1.

(iii) From $X^h = \sum_{j=1}^m \pi_j X = W(\vec{r}) + \sum_{s=1}^k \pi_{i_s} Z(M)$ follows that $X^h = W(\vec{\mu})$ is equivalent to

$$\pi_{i_s} Z(M) = \langle f^{\mu_{i_s}} u_{i_s} \rangle, \quad s = 1, \dots, k,$$

that is, M has no zero row. □

Example 3.6. We refer to Example 3.4 and consider the case $J = \{1, 2, 3\}$ with $\vec{\mu} = (0, 1, 2)$. In that case we have $\vec{r} = (1, 2, 3)$ and therefore $D_{\vec{\mu}} = \text{span}\{u_1, fu_2, f^2u_3\}$. We apply Theorem 3.5 to determine the subspaces X with

$$X_H = W(\vec{r}) \quad \text{and} \quad X^h = W(\vec{\mu}). \quad (3.20)$$

The two matrices M_1 and M_2 that simultaneously satisfy the conditions in Theorem 3.5(ii)-(iii) are

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{with} \quad Z(M_1) = \text{span}\{u_1 + fu_2 + f^2u_3\}$$

and

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{with} \quad Z(M_2) = \text{span}\{u_1 + f^2u_3, fu_2 + f^2u_3\}.$$

The corresponding characteristic subspaces $X_i = W(\vec{r}) + Z(M_i)$, $i = 1, 2$, are

$$X_1 = G = \langle u_1 + fu_2 + f^2u_3 \rangle^c$$

and $X_2 = F = \langle u_1 + fu_2, fu_2 + f^2u_3 \rangle^c$. Hence, $X = G$ and $X = F$ are the only elements of $[W(\vec{r}), W(\vec{\mu})]$ that satisfy (3.20).

4 Extensions

In Theorem 2.5 we have seen that for a given set J a pair of m -tuples $(\vec{r}, \vec{\mu})$ satisfies

$$\vec{\mu} \in \mathcal{L}(\vec{t}) \quad \text{and} \quad \vec{r} = \vec{\mu} + \sum_{i \in J} \vec{e}_i \in \mathcal{L}(\vec{t}) \quad (4.1)$$

only if the inequalities (2.13) hold. In this section we show that (2.13) is sufficient for the existence of such a pair. We use this fact for the construction of characteristic non-hyperinvariant subspaces. Let

$$J = \{i_1, \dots, i_k\} \subseteq \{1, \dots, m\}, \quad i_1 < \dots < i_k, \quad 2 \leq k. \quad (4.2)$$

Set $\vec{t}_J = (t_{i_1}, \dots, t_{i_k})$. Suppose $\vec{\mu}_J = (\tilde{\mu}_{i_1}, \dots, \tilde{\mu}_{i_k}) \in \mathcal{L}(\vec{t}_J)$ and $\vec{\mu} = (\mu_1, \dots, \mu_m) \in \mathcal{L}(\vec{t})$. We call $\vec{\mu}$ an *extension* of $\vec{\mu}_J$ if

$$\mu_{i_s} = \tilde{\mu}_{i_s}, \quad s = 1, \dots, k. \quad (4.3)$$

Let $\mathcal{E}(\vec{\mu}_J)$ be the set of all extensions of $\vec{\mu}_J$. It follows from Lemma 4.1 below that $\mathcal{E}(\vec{\mu}_J)$ is nonempty. Since $\mathcal{E}(\vec{\mu}_J)$ is a sublattice of $\mathcal{L}(\vec{t})$ there exists a maximum element of $\mathcal{E}(\vec{\mu}_J)$, which we call the *maximum extension*.

Suppose $\vec{\mu} \in \mathcal{E}(\vec{\mu}_J)$. We take a closer look at the entries of $\vec{\mu}$. If $i_s \leq j \leq i_{s+1}$ then $\tilde{\mu}_{i_s} \leq \mu_j \leq \tilde{\mu}_{i_{s+1}}$ and

$$t_{i_s} - \tilde{\mu}_{i_s} \leq t_j - \mu_j \leq t_{i_{s+1}} - \tilde{\mu}_{i_{s+1}}. \quad (4.4)$$

Since (4.4) is equivalent to

$$t_j - (t_{i_{s+1}} - \tilde{\mu}_{i_{s+1}}) \leq \mu_j \leq t_j - (t_{i_s} - \tilde{\mu}_{i_s})$$

we obtain

$$\mu_j \leq \min\{t_j - (t_{i_s} - \tilde{\mu}_{i_s}), \tilde{\mu}_{i_{s+1}}\}. \quad (4.5)$$

If $1 \leq j \leq i_1$ then $0 \leq \mu_j \leq \tilde{\mu}_{i_1}$ and $0 \leq t_j - \mu_j$. Hence

$$\mu_j \leq \min\{t_j, \tilde{\mu}_{i_1}\}. \quad (4.6)$$

If $i_k \leq j$ then $\tilde{\mu}_{i_k} \leq \mu_j$ and $t_{i_k} - \tilde{\mu}_{i_k} \leq t_j - \mu_j$, and therefore

$$\tilde{\mu}_{i_k} \leq \mu_j \leq t_j - (t_{i_k} - \tilde{\mu}_{i_k}). \quad (4.7)$$

Lemma 4.1. Assume (4.2). Suppose $\vec{\mu}_J = (\tilde{\mu}_{i_1}, \dots, \tilde{\mu}_{i_k}) \in \mathcal{L}(\vec{t}_J)$, that is,

$$0 \leq \tilde{\mu}_{i_1} \leq \dots \leq \tilde{\mu}_{i_k} \quad \text{and} \quad 0 \leq t_{i_1} - \tilde{\mu}_{i_1} \leq \dots \leq t_{i_k} - \tilde{\mu}_{i_k}. \quad (4.8)$$

Define

$$\mu_j = \begin{cases} \min\{t_j, \tilde{\mu}_{i_1}\} & \text{if } 1 \leq j \leq i_1 \\ \min\{t_j - (t_{i_s} - \tilde{\mu}_{i_s}), \tilde{\mu}_{i_{(s+1)}}\} & \text{if } i_s \leq j \leq i_{s+1}, s = 1, \dots, k-1 \\ t_j - (t_{i_k} - \tilde{\mu}_{i_k}) & \text{if } i_k \leq j \leq m. \end{cases} \quad (4.9)$$

(α) Then $\vec{\mu}$ is the maximum extension of $\vec{\mu}_J$.

(β) If $J \subset I_u$ and

$$0 \leq \tilde{\mu}_{i_1} < \dots < \tilde{\mu}_{i_k} \quad \text{and} \quad 0 < t_{i_1} - \tilde{\mu}_{i_1} < \dots < t_{i_k} - \tilde{\mu}_{i_k}, \quad (4.10)$$

then $\vec{r} = \vec{\mu} + \sum_{i \in J} \vec{e}_i \in \mathcal{L}(\vec{t})$.

Proof. (α) To prove that $\vec{\mu}$ is an extension of $\vec{\mu}_J$ we have to show that the conditions (4.3) and

$$0 \leq \mu_j \leq \mu_{j+1} \quad (4.11)$$

and

$$0 \leq t_j - \mu_j \leq t_{j+1} - \mu_{j+1}, \quad (4.12)$$

$j = 1, \dots, m-1$, are satisfied. We consider different cases.

(i) Case $j = i_s$, $s \in \{1, \dots, k\}$. Then (4.9) yields $\mu_{i_s} = \tilde{\mu}_{i_s}$.

(ii) Case $1 \leq j < i_1$. Then $\mu_j \geq 0$ and $t_j - \mu_j \geq 0$. From $t_j \leq t_{j+1}$ follows

$$\mu_j = \min\{t_j, \tilde{\mu}_{i_1}\} \leq \min\{t_{j+1}, \tilde{\mu}_{i_1}\} = \mu_{j+1}.$$

(I) Case $\mu_j = t_j$. Then $t_j - \mu_j = 0 \leq t_{j+1} - \mu_{j+1}$ such that (4.12) is satisfied.

(II) Case $\mu_j = \tilde{\mu}_{i_1}$. Then $t_{j+1} \geq t_j \geq \tilde{\mu}_{i_1}$, which implies

$$\mu_{j+1} = \min\{t_{j+1}, \tilde{\mu}_{i_1}\} = \tilde{\mu}_{i_1} = \mu_j$$

and $t_{j+1} - \mu_{j+1} \geq t_j - \mu_j$.

(iii) Case $i_k \leq j \leq m$. Then $t_{i_k} \leq t_j$ and $t_j - \mu_j = t_{i_k} - \mu_{i_k}$ imply (4.11) and (4.12), respectively.

(iv) Case $i_s \leq j < i_{(s+1)}$, $s \in \{1, \dots, k-1\}$.

(I) Case $t_j \leq t_{j+1} \leq t_{i_s} + (\tilde{\mu}_{i_{(s+1)}} - \tilde{\mu}_{i_s})$. Then $\mu_j = t_j - (t_{i_s} - \tilde{\mu}_{i_s})$ and $\mu_{j+1} = t_{j+1} - (t_{i_s} - \tilde{\mu}_{i_s})$. Hence $\mu_j \leq \mu_{j+1}$. Moreover, $t_j - \mu_j = t_{j+1} - \mu_{j+1} = t_{i_s} - \tilde{\mu}_{i_s}$.

- (II) Case $t_{i_{(s+1)}} > t_{j+1} \geq t_j \geq t_{i_s} + (\tilde{\mu}_{i_{(s+1)}} - \tilde{\mu}_{i_s})$. Then $\mu_j = \mu_{j+1} = \tilde{\mu}_{i_{(s+1)}}$, which implies (4.12),
- (III) Case $t_j \leq t_{i_s} + (\tilde{\mu}_{i_{(s+1)}} - \tilde{\mu}_{i_s}) \leq t_{j+1}$. Then $\mu_j = t_j - (t_{i_s} - \tilde{\mu}_{i_s}) \leq \tilde{\mu}_{i_{(s+1)}} = \mu_{j+1}$. Hence we obtain

$$t_j - \mu_j = t_{i_s} - \tilde{\mu}_{i_s} \leq t_{j+1} - \tilde{\mu}_{i_{(s+1)}} = t_{j+1} - \mu_{j+1}.$$

From (4.5) - (4.7) we conclude that $\vec{\mu}$ is the maximum element of $\mathcal{E}(\vec{\mu}_J)$.

(β) If $i_s \in I_u$ then the corresponding elementary divisor $\lambda^{t_{i_s}}$ is unrepeated, and therefore

$$t_{(i_s-1)} < t_{i_s} < t_{(i_s+1)}, \quad i = 1, \dots, k. \quad (4.13)$$

We have $\vec{\mu} \in \mathcal{L}(\vec{t})$ and

$$r_j = \begin{cases} \tilde{\mu}_j + 1 & \text{if } j \in J \\ \mu_j & \text{if } j \notin J. \end{cases}$$

Hence in order to prove $\vec{r} \in \mathcal{L}(\vec{t})$ we have to show that

$$\tilde{\mu}_{i_s} < \mu_{(i_s+1)} \quad (4.14)$$

and

$$t_{(i_s-1)} - \mu_{(i_s-1)} < t_{i_s} - \tilde{\mu}_{i_s}, \quad (4.15)$$

$s = 1, 2, \dots, k$. In the case $s = k$ definition (4.9) implies

$$\mu_{i_k+1} = (t_{i_k+1} - t_{i_k}) + \tilde{\mu}_{i_k}.$$

Then (4.13) yields $\mu_{i_k+1} > \tilde{\mu}_{i_k}$. In the case $s < k$ we have

$$\mu_{(i_s+1)} = \min\{t_{(i_s+1)} - (t_{i_s} - \tilde{\mu}_{i_s}), \tilde{\mu}_{i_{(s+1)}}\}.$$

If $\mu_{(i_s+1)} = t_{(i_s+1)} - (t_{i_s} - \tilde{\mu}_{i_s})$ then (4.13) yields (4.14). If $\mu_{(i_s+1)} = \tilde{\mu}_{i_{(s+1)}}$ then (4.14) follows from the strict inequality $\tilde{\mu}_{i_s} < \tilde{\mu}_{i_{(s+1)}}$.

It remains to deal with (4.15). Let $s > 1$. Then $i_{s-1} \leq i_s - 1 < i_s$ implies

$$\mu_{(i_s-1)} = \min\{t_{(i_s-1)} - t_{i_{(s-1)}} + \tilde{\mu}_{i_{(s-1)}}, \tilde{\mu}_{i_s}\}.$$

Suppose $\mu_{(i_s-1)} = t_{(i_s-1)} - t_{i_{(s-1)}} + \tilde{\mu}_{i_{(s-1)}}$. Then $t_{i_{(s-1)}} - \tilde{\mu}_{i_{(s-1)}} < t_{i_s} - \tilde{\mu}_{i_s}$ implies

$$\begin{aligned} t_{(i_s-1)} - \mu_{(i_s-1)} &= t_{(i_s-1)} - [t_{(i_s-1)} - t_{i_{(s-1)}} + \tilde{\mu}_{i_{(s-1)}}] = \\ &= t_{i_{(s-1)}} - \tilde{\mu}_{i_{(s-1)}} < t_{i_s} - \tilde{\mu}_{i_s}. \end{aligned}$$

Suppose $\mu_{(i_s-1)} = \tilde{\mu}_{i_s}$. Then (4.13) implies

$$t_{(i_s-1)} - \mu_{(i_s-1)} = t_{(i_s-1)} - \tilde{\mu}_{i_s} < t_{i_s} - \tilde{\mu}_{i_s}.$$

Let $s = 1$. Then $\mu_{(i_1-1)} = \min\{t_{(i_1-1)}, \tilde{\mu}_{i_1}\}$. If $\mu_{(i_1-1)} = \tilde{\mu}_{i_1}$, then

$$t_{(i_1-1)} - \mu_{(i_1-1)} < t_{i_1} - \tilde{\mu}_{i_1} \quad (4.16)$$

follows from (4.13). If $\mu_{(i_1-1)} = t_{(i_1-1)}$ then the strict inequality $0 < t_{i_1} - \tilde{\mu}_{i_1}$ in (4.10) implies (4.16). \square

We note without proof that the minimum element of $\mathcal{E}(\vec{\mu}_J)$ is given by

$$\mu_j = \begin{cases} \max\{0, t_j - (t_{i_1} - \tilde{\mu}_{i_1})\} & \text{if } 1 \leq j \leq i_1 \\ \max\{t_j - (t_{i_{(s+1)}} - \tilde{\mu}_{i_{(s+1)}}), \tilde{\mu}_{i_s}\} & \text{if } i_s \leq j \leq i_{s+1}, s = 1, \dots, k-1 \\ \tilde{\mu}_{i_k} & \text{if } i_k \leq j \leq m. \end{cases} \quad (4.17)$$

The next theorem provides an existence result. It shows that to a given admissible set J there exists a characteristic non-hyperinvariant subspace X such that $J(X) = J$.

Theorem 4.2. *Assume*

$$J = \{i_1, \dots, i_k\} \subseteq I_u, i_1 < \dots < i_k, 2 \leq k. \quad (4.18)$$

Suppose $\vec{\mu}_J = (\tilde{\mu}_{i_1}, \dots, \tilde{\mu}_{i_k}) \in \mathcal{L}(\vec{t}_J)$ and let $\vec{\mu} = (\mu_1, \dots, \mu_m) \in \mathcal{L}(\vec{t})$ be the maximum extension of $\vec{\mu}_J$. Set $\vec{r} = \vec{\mu} + \sum_{i \in J} \vec{e}_i$. Then the following statements are equivalent.

- (i) The entries of $\vec{\mu}_J$ satisfy the inequalities $0 \leq \tilde{\mu}_{i_1} < \dots < \tilde{\mu}_{i_k}$ and the strict inequalities $0 < t_{i_1} - \tilde{\mu}_{i_1} < \dots < t_{i_k} - \tilde{\mu}_{i_k}$.
- (ii) There exists a characteristic non-hyperinvariant subspace X with hyperinvariant frame $(X_H, X^h) = (W(\vec{r}), W(\vec{\mu}))$.

Proof. The implication (i) \Rightarrow (ii) follows from Theorem 3.3 and Lemma 4.1(ii) and the implication (ii) \Rightarrow (i) is consequence of Theorem 2.5. \square

In the case of the maximum extension $\vec{\mu}$ of $\vec{\mu}_J$ one can give a concise description of the subspaces in $[W(\vec{r}), W(\vec{\mu})]$ in Theorem 3.3.

Theorem 4.3. Assume (4.18) and

$$0 \leq \mu_{i_1} < \cdots < \mu_{i_k} \quad \text{and} \quad 0 < t_{i_1} - \mu_{i_1} < \cdots < t_{i_k} - \mu_{i_k}. \quad (4.19)$$

Let $\vec{\mu} = (\mu_1, \dots, \mu_m)$ be the maximum extension of $\vec{\mu}_J = (\mu_{i_1}, \dots, \mu_{i_k})$ and let $\vec{r} = \vec{\mu} + \sum_{i \in J} \vec{e}_i$. Let $Z = \text{span}\{z_1, \dots, z_q\}$ be a q -dimensional subspace of

$$D_{\vec{\mu}_J} = \text{span}\{f^{\mu_{i_1}} u_{i_1}, \dots, f^{\mu_{i_k}} u_{i_k}\}. \quad (4.20)$$

If $X = W(\vec{r}) \oplus Z$ satisfies $X^h = W(\vec{\mu})$ then $X = \langle z_1, \dots, z_q \rangle^c$.

Proof. It is obvious that $Z^c \subseteq X$. Because of $Z \subseteq X$ the converse inclusion $X \subseteq Z^c$ is equivalent to

$$W(\vec{r}) = \langle f^{r_1} u_1, \dots, f^{r_m} u_m \rangle \subseteq Z^c.$$

Set $\mathfrak{Z} = \{z_1, \dots, z_q\}$. To check that $f^{r_j} u_j \in Z^c$ we separately deal with different cases of j . In each of the cases (i) - (iv) below we choose suitable automorphisms $\alpha \in \text{Aut}(f, V)$ such that $\alpha z = z + f^{r_j} u_j$ for some $z \in \mathfrak{Z}$. Then $f^{r_j} u_j \in \langle z \rangle^c \subseteq Z^c$. The assumption $X^h = W(\vec{\mu})$ implies that for each $j \in J$ there exists an element $z \in \mathfrak{Z}$ such that $\pi_j z \neq 0$. Let $\alpha(u_j, u'_j)$ denote the automorphism that exchanges the generator u_j by u'_j . Recall that $r_j = 1 + \mu_j$ if $j \in J$ and $r_j = \mu_j$ if $j \notin J$.

- (i) Case $j = i_s$, $s \in \{1, \dots, k\}$. If $z \in \mathfrak{Z}$ satisfies $\pi_{i_s} z \neq 0$ then $\alpha = \alpha(u_{i_s}, u_{i_s} + f u_{i_s})$ yields $\alpha z = z + f^{1+\mu_{i_s}} u_{i_s}$.
- (ii) Case $1 \leq j < i_1$. If $\mu_j = r_j = t_j$ then it is obvious that $f^{r_j} u_j = 0 \in Z^c$. If $\mu_j = r_j = \mu_{i_1}$ then $i_1 \in I_u$ implies $e(u_j) < e(u_{i_1})$ and $e(u_{i_1}) = e(u_{i_1} + u_j)$. Hence, if $z \in \mathfrak{Z}$ satisfies $\pi_{i_1} z \neq 0$ then $\alpha = \alpha(u_{i_1}, u_{i_1} + u_j)$ yields $\alpha z = z + f^{\mu_{i_1}} u_j = z + f^{r_j} u_j$.
- (iii) Case $m \geq j > i_k$. Then $t_j > t_{i_k}$ and $\mu_j = t_j - t_{i_k} + \mu_{i_k}$. We have $e(u_{i_k}) = e(u_{i_k} + f^{t_j - t_{i_k}} u_j)$. If $z \in \mathfrak{Z}$ satisfies $\pi_{i_k} z \neq 0$ then $\alpha = \alpha(u_{i_k}, u_{i_k} + f^{t_j - t_{i_k}} u_j)$ yields $\alpha z = z + f^{\mu_{i_k} + (t_j - t_{i_k})} u_j$.
- (iv) Case $i_s < j < i_{(s+1)}$, $s \in \{1, \dots, k-1\}$.
 - (I) Case $t_j \leq t_{i_s} + (\mu_{i_{(s+1)}} - \mu_{i_s})$. Then $\mu_j = t_j - (t_{i_s} - \mu_{i_s})$. If $z \in \mathfrak{Z}$ satisfies $\pi_{i_s} z \neq 0$ then $\alpha = \alpha(u_{i_s}, u_{i_s} + f^{t_j - t_{i_s}} u_j)$ yields $\alpha z = z + f^{\mu_{i_s} + t_j - t_{i_s}} u_j$.
 - (II) Case $t_j \geq t_{i_s} + (\mu_{i_{(s+1)}} - \mu_{i_s})$. Then $\mu_j = \mu_{i_{(s+1)}}$. If $z \in \mathfrak{Z}$ satisfies $\pi_{i_{(s+1)}} z \neq 0$ then $\alpha = \alpha(u_{i_{(s+1)}}, u_{i_{(s+1)}} + u_j)$ yields $\alpha z = z + f^{\mu_{i_{(s+1)}}} u_j$.

□

Corollary 4.4. *Assume (4.18) and (4.19). Let*

$$z = f^{\mu_{i_1}} u_{i_1} + \cdots + f^{\mu_{i_k}} u_{i_k}. \quad (4.21)$$

If $\vec{\mu}$ is the maximum extension of $\vec{\mu}_J = (\mu_{i_1}, \dots, \mu_{i_k})$ and $\vec{r} = \vec{\mu} + \sum_{s=1}^k \vec{e}_{i_s}$ then $X(z) = W(\vec{r}) + \text{span}\{z\}$ is a characteristic non-hyperinvariant subspace and $X(z) = \langle z \rangle^c$.

Proof. We have $Z = \text{span}\{z\} = Z(M)$ with

$$M = \begin{pmatrix} e & 0_{k \times (k-1)} \end{pmatrix} \quad \text{and} \quad e = (1, 1, \dots, 1)^T.$$

Theorem 3.5 implies $X^h = W(\vec{\mu})$. Hence we can apply Theorem 4.3. Moreover, $X_H = W(\vec{r})$. Therefore X is not hyperinvariant. □

Example 4.5. Let (V, f) be given by (1.11). Referring to Example 3.4 we consider the case $J = \{1, 3\} \subseteq I_u = \{1, 2, 3\}$ with $\vec{\mu}_J = (\mu_1, \mu_3) = (0, 2)$. Two extensions $\vec{\mu}$ of $\vec{\mu}_J$ satisfy $\vec{r} = \vec{\mu} + (1, 0, 1) \in \mathcal{L}(\vec{t})$, namely $\vec{\mu} = \vec{\mu}_1 = (0, 1, 2)$ and $\vec{\mu} = \vec{\mu}_2 = (0, 2, 2)$. The corresponding triples \vec{r} are $\vec{r}_1 = (1, 1, 3)$ and $\vec{r}_2 = (1, 2, 3)$. We have $\vec{\mu}_1 \preceq \vec{\mu}_2$. Thus $\vec{\mu}_2$ is the maximum extension in \mathcal{E}_J . Define $z = u_1 + f^2 u_3$ according to (4.21). Then

$$X(z) = W(\vec{r}_2) + \text{span}\{z\} = \langle f^2 u_2, f^3 u_3, u_1 + f^2 u_3 \rangle = \langle z \rangle^c$$

is a characteristic non-hyperinvariant subspace of V . In the case of $\vec{\mu} = \vec{\mu}_1$ we recall (3.16) and note that $X = W(\vec{r}_1) + \text{span}\{z\}$ is not the characteristic hull of a single vector.

In [4] we studied invariant subspaces that are the characteristic hull of a single vector. Using a decomposition lemma due to Baer [6] we proved the following result, which yields part of Corollary 4.4.

Theorem 4.6. *For a given nonzero $z \in V$ there exists a generator tuple $U = (u_1, \dots, u_m)$ such that z can be represented in the form*

$$z = f^{\mu_{\rho_1}} u_{\rho_1} + \cdots + f^{\mu_{\rho_k}} u_{\rho_k}, \quad (4.22)$$

with

$$0 \leq \mu_{\rho_1} < \cdots < \mu_{\rho_k} \quad \text{and} \quad 0 < t_{\rho_1} - \mu_{\rho_1} < \cdots < t_{\rho_k} - \mu_{\rho_k}. \quad (4.23)$$

The following statements are equivalent.

- (i) *The subspace $X = \langle z \rangle^c$ is not hyperinvariant.*
- (ii) *At least two of the generators u_{ρ_i} in (4.22) are unrepeated.*

The assumptions (4.18) and (4.19) in Corollary 4.4 imply that in (4.21) all generators u_{i_s} , $s = 1, \dots, k$, are unrepeated and that (4.23) holds. Hence it follows from Theorem 4.6 that the characteristic subspace $X(z) = \langle z \rangle^c$ is not hyperinvariant.

We reexamine Shoda's theorem. Using Corollary 4.4 or Theorem 4.6 we refine the implication (ii) \Rightarrow (i) in Theorem 1.1.

Corollary 4.7. *Let λ^R and λ^S be unrepeated elementary divisors of f such that $R + 1 < S$. Let u and v be generators of (V, f) with $e(u) = R$ and $e(v) = S$. If the integers s and q satisfy*

$$0 \leq s < q \quad \text{and} \quad 0 < R - s < S - q$$

then $X = \langle f^s u + f^q v \rangle^c$ is a characteristic subspace of V that is not hyperinvariant.

5 Concluding remarks

From Theorem 1.2 one can deduce properties of the lattice of $\text{Hinv}(V, f)$. It is known [9] that $\text{Hinv}(V, f)$ is self-dual in the sense that there exists a bijective map $\Lambda : \text{Hinv}(V, f) \rightarrow \text{Hinv}(V, f)$ such that

$$\Lambda(W + Y) = \Lambda(W) \cap \Lambda(Y) \quad \text{and} \quad \Lambda(W \cap Y) = \Lambda(W) + \Lambda(Y)$$

for all $W, Y \in \text{Hinv}(V, f)$. It is not difficult to show (see [12, p. 343] that $\vec{r} \in \mathcal{L}(\vec{t})$ if and only if $\overrightarrow{t - r} \in \mathcal{L}(\vec{t})$. Hence, if $\vec{r} \in \mathcal{L}(\vec{t})$ and

$$W(\vec{r}) = f^{r_1} V \cap V[f^{t_1 - r_1}] + \dots + f^{r_m} V \cap V[f^{t_m - r_m}] \in \mathcal{L}(\vec{t})$$

then $\Lambda(W(\vec{r}))$ is given by

$$\Lambda(W(\vec{r})) = W(\overrightarrow{t - r}) = f^{t_1 - r_1} V \cap V[f^{r_1}] + \dots + f^{t_m - r_m} V \cap V[f^{r_m}].$$

For the moment it is an open problem whether the lattice $\text{Chinv}(V, f)$ is self-dual, and it remains to clarify the lattice structure of $\text{Chinv}(V, f)$. A useful tool for such an investigation will be the concept of hyperinvariant frame, which we introduced in this paper.

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